## ON WILSON SURFACES IN TOPOLOGICAL CHERN-SIMONS INTERACTION IN (3+1) DIMENSIONS

JUNGJAI LEE\*

Department of Physics, Daejin University, Mt.11-1, Seondan, Pocheon, Kyeonggi, Korea

## ABSTRACT

We comment on the regularization of the expectation values of Wilson surfaces for the bosonic string in the (3+1) dimensions. We analyze the singular behaviors of propagator for the Chern-Simons action with the additional higher order terms.

<sup>\*</sup> e-mail address: jjlee@road.daejin.ac.kr

<sup>†</sup> This work was supported by the Dae-Jin University Research Grants in 1996.

In paper [1], Polyakov explicitly showed that the charged particles reverse their statistics in the (2+1) dimensional abelian gauge theory with the topological Chern-Simons term. Polyakov's spin factor, which is given as the expectation values of Wilson line operator, plays a central role in this phenomenon. After this work, Xavier Fustero and et' al [2] showed that the naive extension of Polyakov's construction to the (3+1) dimensional case led to the transmutation between bosonic and fermionic one-dimensional structures. They established this fact through the argument that the Dirac algebra of  $e_{\mu}$  fields may be represented in terms of a one-parameter family of Pauli matrices, i.e. the spin chain operator  $\sigma(t)$ , which provides the needed ingredient to reproduce the propagator for the fermionic string.

In the (2+1) dimensional Chern-Simons theory we need to introduce a framing prescription for the expectation values of Wilson line operators [3]. This is due to the fact that the metric is engaged in the gauge fixing procedure while those operators are topological in character. However, in the (3+1)dimensional case, the extension is a sort of BF theory. According to the illustration in Ref.[4], we have the antisymmetric B field in addition to the connection field A, the question of framing does not arise in the analogous calculations. The result is, therefore, finite and unambiguous.

In this paper, we will show that if we try to calculate the expectation values of the Wilson surface operator in (3+1) dimensional bosonic string theory the careful regularization must be needed. To do this, we will give the two definitions for the expectation values of the Wilson surface operator and discuss their singular behaviors near the boundary of the string world sheets. Next the effect of the higher order terms(kinetic terms) in the action will be analyzed. We will also investigate the singular behaviors of spin factors in the time like gauge, which is independent of the metric, and compare with that in the covariant gauge. Finally we will discuss a problem with the gauge invariance of the antisymmetric 2-form field  $B_{\mu\nu}$ .

According to Ref.[2], the spin factor  $\Phi(S, C)$  between the initial and final spatial configurations  $P_i$  and  $P_f$  of the string is given as

$$exp(i\Phi(S,C))$$

$$= \langle exp(i\int_{S} d\sigma^{\mu\nu} B_{\mu\nu}) exp(i\oint_{C} dx^{\mu} A_{\mu}) \rangle$$

$$= \int DADBexp(-i\int_{S} d^{4}x \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} \partial_{\rho} A_{\sigma}) exp(i\int_{S} d\sigma^{\mu\nu} B_{\mu\nu}) exp(i\oint_{C} dx^{\mu} A_{\mu})$$
(1)

Here the S and C in  $\Phi(S, C)$  denote the world sheets and its border respectively, and we regard the spin factor  $\Phi(S, C)$  as the expectation value of the Wilson surface operator in the BF theory.

In above path integral (1), we considered a (3+1)dimensional bosonic-string classical system in the electromagnetic interactions described by the gauge potential  $A_{\mu}(x)$  and the antisymmetric background field  $B_{\mu\nu}(x)$ . These fields are coupled through the topological Chern-Simons term

$$\int d^4x \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} \partial_\rho A_\sigma \tag{2}$$

which is invariant under the gauge transformations with the gauge parameters  $\Lambda$  and  $\xi_{\mu}$ ;

$$\delta A_{\mu} = \partial_{\mu} \Lambda, \tag{3}$$

$$\delta B_{\mu\nu} = \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu}. \tag{4}$$

The integrals (1) over the  $A_{\mu}$  and  $B_{\mu\nu}$  fields lead to the spin factor

$$\Phi(S,C) = -\frac{i}{4} \int_{S_C} d\sigma_{\mu\nu}(y) \int_{S'_C} d\sigma_{\mu\nu}(y') \delta(y-y')$$

$$= -\frac{i}{4\pi^2} \oint_C dy_{\mu} \int_{S'_C} d\sigma_{\lambda\rho}(y') \epsilon_{\mu\nu\lambda\rho} \partial_{\nu} \frac{1}{|y-y'|^2} \tag{5}$$

where  $S'_{C}$  is the second arbitrary surface with border C.

This expression for the spin factor (5) equals to the writhe of ribbon as the string world sheet [5] [6]. The writhe depends on the metric, this dependence enters the propagator via the gauge fixing conditions  $\partial_{\mu}A^{\mu} = 0$  and  $\partial_{\mu}B^{\mu\nu} = 0$ . The phase factor  $\Phi(S, C)$  is singular, since C must tend to the border of  $\sigma$ , therefore the careful regularization process is needed. First we introduce the framing

$$\widetilde{\Phi}(S,C) \equiv \lim_{\epsilon \to 0} \left( -\frac{i}{4\pi^2} \oint_C dy_\mu \int_{S'C} d\sigma_{\lambda\rho}(y') \epsilon_{\mu\nu\lambda\rho} \partial_\nu \frac{1}{|y-y'-\epsilon \hat{n}(y)|^2} \right)$$

$$= L(S,C,\hat{n}),$$
(6)

where  $\hat{n} = (n_1, n_2, e, f)$ ;  $n_{1\mu}$  and  $n_{2\mu}$  are two normal vectors orthogonal to the both  $e_{\mu}$  and  $f_{\mu}$  which are the unit tangent vectors to the closed path C in (3+1) dimensions. The  $\hat{n}$  defines a framing of the surface S with its border C. In the equation (6), L is the linking number, a topological quantity, which measure the revolution times of the closed path C around the sheet S in four dimensions.

Let us compare the two expressions for the spin factor. The expression (5) for the spin factor is metric dependent and framing independent, while on the other the expression (6) framing dependent and metric independent, so we can ask which definition is correct. This metric dependence in eq. (4) originates in the short distance singularities of the propagator which enters via the gauge fixing condition at the quantum level.

To clarify the singular behavior of the propagator, we add the higher order terms to the action (2). This work is physically meaningful as higher order terms are metric dependent but frame independent. So we take the following action for gauge fields from the effective field theory of quantized strings [7]:

$$\Gamma = \int d^4x (\alpha_1 F^{\mu\nu} F_{\mu\nu} + \alpha_2 H^{\mu\nu\rho} H_{\mu\nu\rho} + \alpha_3 \epsilon^{\mu\nu\rho\sigma} A_{\mu} \partial_{\nu} B_{\rho\sigma}), \tag{7}$$

where  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  are constants. Here

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu},$$

$$H_{\mu\nu\rho} = \partial_{\mu}B_{\nu\rho} + cyclic \ permutations \ of \ \mu, \ \nu, \ \rho.$$
(8)

In the covariant (Feynman) gauge we obtain the propagator

$$D_{\mu\nu\rho}(x,y) = D_{\mu\nu\rho}(|x-y|) = \langle A_{\mu}(x)B_{\nu\rho}(y) \rangle,$$

$$D_{\mu\nu\rho}(r) = \frac{1}{32\pi^{2}\alpha_{1}\alpha_{3}} \epsilon^{\mu\nu\rho\sigma} \partial_{\sigma} (\frac{1}{\mu^{2}r^{2}} - \frac{1}{\mu r} K_{1}(\mu r))$$

$$= \frac{1}{32\pi^{2}\alpha_{1}\alpha_{3}} \epsilon^{\mu\nu\rho\sigma} \hat{r}_{\sigma} (\frac{-2}{\mu^{2}r^{3}} - \frac{1}{r} K_{2}(\mu r)),$$
(9)

where r = |x - y|,  $\mu^2 = \frac{\alpha_3^2}{\alpha_2}$ ;  $\mu$  is a topological mass,  $K_1(x)$  and  $K_2(x)$  are the modified Bessel functions. Though we can get three propagators in this theory, we will consider only  $\langle A_{\mu}B_{\nu\rho} \rangle$  propagator since we are concerned only with the topological nature of Chern-Simons term, i.e. the statistics of the theory. To investigate how the presence of the  $F_{\mu\nu}F^{\mu\nu}$  and  $H_{\mu\nu\rho}H^{\mu\nu\rho}$  terms modify the fields close to the border C, consider

$$A_{\mu} = \int_{S_{C}} d\sigma_{\nu\rho} D_{\mu\nu\rho}(r) + Coulomb \ type \ singularity$$

$$= \frac{1}{32\pi^{2}\alpha_{1}\alpha_{3}} \int_{S_{C}} d\sigma_{\nu\rho} \epsilon^{\mu\nu\rho\sigma} \hat{r}_{\sigma} (\frac{-2}{\mu^{2}r^{3}} - \frac{1}{r}K_{2}(\mu r)) + Coulomb \ type \ singularity$$

$$= \frac{1}{32\pi^{2}\alpha_{1}\alpha_{3}} \int_{S_{C}} d\sigma_{\nu\rho} \epsilon^{\mu\nu\rho\sigma} \hat{r}_{\sigma} f(r) + Coulomb \ type \ singularity,$$

$$(10)$$

where the Coulomb type singularity comes from the  $\langle A_{\mu}A_{\nu} \rangle$  propagator. Since the function f(r) behaves as follows  $f(r) \sim \frac{2}{\mu^2 r^3}$  for  $r \gg \frac{1}{\mu}$  and  $f(r) \sim \frac{\mu}{r}$  for  $r \ll \frac{1}{\mu}$  the field strength is smooth and concentrated to a region  $\sim \frac{1}{\mu}$  around the border C. The string-like magnetic flux present in the topological BF theory is distributed over a region with size  $\sim r \ll \frac{1}{\mu}$  near the border of the open string world sheet in (3+1) dimension. Let the border of  $S_C$  be  $C_1$  and the border of  $S'_C$ be  $C_2$ , if  $B_{\mu\nu}(x)$  is sufficiently far away from boundary we obtain the topological quantity L where L is the linking number of the two curves  $C_1$  and  $C_2$ . If the curves are within a distance  $\sim \frac{1}{\mu}$ , the magnetic flux are overlapped, therefore the statistical interpretation for the phase fails. By adding the  $F_{\mu\nu}F^{\mu\nu}$  and  $H_{\mu\nu\rho}H^{\mu\nu\rho}$  terms to the action, the expectation value of Wilson surface can be unambiguously defined; the spin factor is metric dependent but framing independent. In the limit  $\mu \to \infty$  where gauge quanta become heavy, the long range interaction parts can be ignored, the phase factor leads to the writhe.

Since in the pure theory without higher order terms the metric enter the propagator via gauge fixing, now we consider the gauge fixing without the metric dependence. In time like gauge,  $A_0 = 0$  and  $B_{0\mu} = 0$ , we obtain the spin factor,

$$\Phi^{0}(S,C) = \oint_{C} dx_{\mu} \int_{\sigma'C} d\sigma_{\nu\rho} \epsilon_{\mu\nu\rho0} \Theta(x^{0} - y^{0}) \delta^{3}(x^{i} - y^{i}), \tag{11}$$

where i=1,2,3 and  $\Theta$  is the step function. Though  $\Phi^0(S,C)$  does not depend on the metric but it is ill-defined (contained  $\delta^3(0)$ ) and should be regularized. If we use the regularization  $\delta(x) \to \frac{1}{\sqrt{2\pi\epsilon}} exp(-x^2/\epsilon)$  as in Ref.[1], such regularization will give the writhe and hence the metric dependence of  $\Phi^0$  inevitable.

In discussion, so far we have analyzed the regularizations of the expectation values of Wilson surfaces. However it is required the more intimate discussion about the gauge invariance of this theory, because the Wilson surface operator with boundaries may not be invariant under the gauge transformation(4) with respect to two form  $B_{\mu\nu}$ . In order to deal with a problem of gauge invariance more intimately, first, recall that in this paper we have introduced two kinds of wilson surface(line) operators:

$$W_B = exp(i \int_S d\sigma^{\mu\nu} B_{\mu\nu}) \tag{12}$$

$$W_A = \exp(i \oint_C dx^\mu A_\mu) \tag{13}$$

As it was shown in Ref.[4], the expectation value of either one of these equal to 1 with respect to pure CS action. The non-trivial expectation values occur from  $\langle W_A W_B \rangle$  where the expectation value is, of course, taken with repect to the pure action(2).

In the fundamental string theory with the open bosonic sector, the antisymmetric tensor gauge invariance must be act on the photon as well. The correct gauge transformation is as follows:

$$\delta B_{\mu\nu} = \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu}$$

$$\delta A_{\mu} = -2\xi_{\mu}$$
(14)

At string world sheet level this appear since a surface term for the variation of  $B_{\mu\nu}$  must be cancelled by variation of the open string vector. And there, of course, exists still the usual gauge invariance of vector  $A_{\mu}$  independently. We can, however, see that under this gauge transformation the pure action is not invariant and hence some correct form of the pure action required. Fortunately we can find the correct gauge invariant form of the action under the gauge transformation (14) as the following modified form:

$$S_M = \int d^4x \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} (\partial_\rho A_\sigma + \frac{1}{2} B_{\rho\sigma})$$
 (15)

It is easy to see that the above action(15) is manifestly invariant under the gauge transformation (14). By replacing dA of dA + B in form notation, it is possible to construct the modified action with higher order terms invariant under the gauge transformation (14). In spite of this change of symmetry, we can still show that the singular behavior of propagator for the system with gauge symmetry (3)(4) is identical to that of the system with gauge symmetry (15). We think, however, as to the modified gauge symmetry the more detail investigations will be needed [8].

On the other hand, in closed string theory we can give the definition of Wilson surface for a closed surface S and a loop C as that of Ref.[4], however the interaction of Wilson line(surface) is invisible because the photon field  $A_{\mu}$  is not appeared explicitly in the closed bosonic string effective theory [7].

In conclusion, we have given the two definitions eq.(6) and eq.(5),(7) of the expectation values of wilson surfaces operator through the different regularizations

which are the ribbon splitting and the gauge invariant higher order term adding. The regularization by adding gauge invariant higher order terms is manifestly gauge invariant and the ribbon splitting is also gauge invariant, so we think that the gauge invariance can not define the regularization uniquely.

The difference between the two definitions eq.(6) and eq.(5)(7) can be resolved by noting the fact that the limit  $\epsilon \to 0$  in eq.(6) is not smooth, since we have the relation W(writhe) = L(link) - T(torsion) for general twisted ribbon. The definition(6) is a topological invariant whose values depend on the framing  $\hat{n}$ , while the definition(5)(7) by adding the higher oder terms lead to the writhe whose values depend on the metric. We think the existence of two definitions (6), (5)(7) originates in the singular behavior of propagator in the short distance, and the reason why the propagator is singular is due to the fact that the curve on which  $A_{\mu}(x)$  field is defined must tend to the border of the string world sheet on which the antisymmetric  $B_{\mu\nu}$  field is defined.

The author is grateful to Jinho Cho and Hyeonjoon Shin for many helpful discussions. The author also thanks Won-Tae Kim for some remarks on regularizations.

## REFERENCES

- [1] A. M. Polyakov, Mod. Phys. Lett. **A3** (1988) 325.
- [2] Xavier Fustero, Rodolfo Gambini and Antoni Trias, Phys. Rev. Lett. 62 (1989) 1964.
- [3] T. H. Hansson, A. Karlhede and M. Roček, Phys. Lett. **B225** (1989) 92.
- [4] M. Blau and G. Thompson, Ann. Phys. (NY) **205** (1991) 130.
- [5] J. Grunberg, T. H. Hansson, A. Karlhede and U. Lindstrom, Phys. Lett. B218 (1989) 321.
- [6] M. D. Frank-Kamenetskij and A. V. Vologodskij, Sov. Phys. Usp. 24 (1981) 679.
- [7] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **B158** (1985) 316; E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **B160** (1985) 69.
- [8] J. H. Cho and J. J. Lee in preparation.